- The weakness of the Backward-Difference method results from the fact that the local truncation error has one of order $O\left(\left(\Delta x\right)^2\right)$, and another of order $O\left(\Delta t\right)$. This requires that time intervals be made much smaller than the x-axis intervals.
- It would clearly desirable to have a procedure with local truncation error of order $O\left(\left(\Delta x\right)^2 + \left(\Delta t\right)^2\right)$.
- The first step in this direction is to use central difference equation for time derivative which has a truncation error of order $O\left(\left(\Delta t\right)^2\right)$. But, unfortunately this method has serious stability problems.

Crank-Nicolson Method

This method is derived by averaging the Forward-Difference method at the *n*th step in *t*,

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} - \alpha^2 \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2} = 0$$

which has the local truncation error,

$$\tau_F = \frac{\Delta t}{2} \frac{\partial^2 u}{\partial t^2} + O\left((\Delta t)^2\right) + O\left((\Delta x)^2\right)$$

and the Backward-Difference method at the (n+1)th step in t,

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} - \alpha^2 \frac{u_{i+1}^{n+1} - 2u_i^{n+1} + u_{i-1}^{n+1}}{(\Delta x)^2} = 0$$

which has the local truncation error,

$$\tau_B = -\frac{\Delta t}{2} \frac{\partial^2 u}{\partial t^2} + O\left((\Delta t)^2\right) + O\left((\Delta x)^2\right)$$

Note that the averaged-difference method,

$$\frac{u_i^{n+1}-u_i^n}{\Delta t}-\frac{\alpha^2}{2}\left[\frac{u_{i+1}^n-2u_i^n+u_{i-1}^n}{(\Delta x)^2}+\frac{u_{i+1}^{n+1}-2u_i^{n+1}+u_{i-1}^{n+1}}{(\Delta x)^2}\right]=0$$

has local truncation error of order $O\left((\Delta x)^2+(\Delta t)^2\right)$. This is known as the **Crank-Nicolson** method. Recalling that,

$$u(0,t)=u(l,t)=0, \quad t>0, \quad \text{and} \quad u(x,0)=f(x), \quad 0\le x\le l$$

this method can be written in the matrix form as,

$$Au^{n+1}=Bu^n$$

where,

$$u^n = \left(u_1^n \quad u_2^n \quad \dots \quad u_{m-1}^n \right)^t \qquad \lambda = \alpha^2 \frac{\Delta t}{(\Delta x)^2}$$

and the matrices A and B are given by,

$$A = \begin{bmatrix} (1+\lambda) & -\frac{\lambda}{2} & 0 & \cdots & 0 \\ -\frac{\lambda}{2} & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & -\frac{\lambda}{2} \\ 0 & \cdots & 0 & -\frac{\lambda}{2} & (1+\lambda) \end{bmatrix}$$

and,

$$B = \begin{bmatrix} (1-\lambda) & \frac{\lambda}{2} & 0 & \cdots & 0 \\ \frac{\lambda}{2} & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \frac{\lambda}{2} \\ 0 & \cdots & 0 & \frac{\lambda}{2} & (1-\lambda) \end{bmatrix}$$

- The matrix A is **positive definite**, **strictly diagonally dominant** and **tridiagonal**. So, **Crout factorization** (for rather small systems) or **SOR** method (for large systems) can be used to solve the system.
- The Crank-Nicolson method is **unconditionally stable** and has local truncation error $O\left((\Delta x)^2+(\Delta t)^2\right)$

Example

Use the Crank-Nicolson method with $\Delta x=0.1$ and $\Delta t=0.01$ to approximate the solution to the heat equation

$$\frac{\partial u}{\partial t}(x,t)-\frac{\partial^2 u}{\partial x^2}(x,t)=0, \quad 0<x<1, \quad 0<t$$

subject to the constraints

$$u(0,t) = u(1,t) = 0, \quad 0 < t, \quad u(x,0) = \sin \pi x, \quad 0 \le x \le 1$$

Solution

The following table represents the results for the Crank-Nicolson method.

x_i	u_i^{50}	exact $u(x_i, 0.5)$	$\lvert u_i^{50} - u(x_i, 0.5)\rvert$
0.0	0	0	
0.1	0.00230512	0.00222241	8.271×10^{-5}
0.2	0.00438461	0.00422728	1.573×10^{-4}
0.3	0.00603489	0.00581836	2.165×10^{-4}
0.4	0.00709444	0.00683989	2.546×10^{-4}
0.5	0.00745954	0.00719188	2.677×10^{-4}
0.6	0.00709444	0.00683989	2.546×10^{-4}
0.7	0.00603489	0.00581836	2.165×10^{-4}
0.8	0.00438461	0.00422728	1.573×10^{-4}
0.9	0.00230512	0.00222241	8.271×10^{-5}
1.0	0	0	

- Recall that the Forward-Difference method gave dramatically poor results for this choice of Δt and Δx , but the Backward-Difference method gave results that were accurate to about 2×10^{-3} for entries in the middle of the table.
- The results in the table indicate the increase in accuracy of the Crank-Nicolson method over the Backward-Difference method.

Solving Hyperbolic Partial Differential Equations

$$\frac{\partial^2 u}{\partial t^2}(x,t) - \alpha^2 \frac{\partial^2 u}{\partial x^2}(x,t) = 0, \quad 0 < x < l, \quad t > 0$$

$$u(0,t) = u(l,t) = 0, \quad \text{for} \quad t > 0$$

$$u(x,0) = f(x), \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = g(x), \quad \text{for} \quad 0 \le x \le l$$

Using central approximation for time and spatial derivatives, we have,

$$\frac{u_i^{n+1}-2u_i^n+u_i^{n-1}}{(\Delta t)^2}-\alpha^2\frac{u_{i+1}^n-2u_i^n+u_{i-1}^n}{(\Delta x)^2}=0 \qquad i=1,2,\ldots,m-1$$

This equation can be rewritten as,

$$u_i^{n+1}=2\left(1-\lambda^2\right)u_i^n+\lambda^2\left(u_{i+1}^n+u_{i-1}^n\right)-u_i^{n-1} \qquad i=1,2,\ldots,m-1$$

where, $\lambda=(\alpha\Delta t)/\Delta x$. Recalling that,

$$u(0,t)=u(l,t)=0, \quad t>0, \quad \text{and} \quad u(x,0)=f(x), \quad 0\le x\le l$$

The discretized equation can be written in the matrix form as,

$$u^{n+1}=Au^n-u^{n-1}$$

where,

$$u^n = \begin{pmatrix} u_1^n & u_2^n & \ldots & u_{m-1}^n \end{pmatrix}^t$$

and,

$$A = \begin{bmatrix} 2(1-\lambda^2) & \lambda^2 & 0 & \cdots & 0 \\ \lambda^2 & 2(1-\lambda^2) & \lambda^2 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \lambda^2 \\ 0 & \cdots & 0 & \lambda^2 & 2(1-\lambda^2) \end{bmatrix}$$

The discretized form of the wave equation imply that the (n+1)th time step requires values from the nth and (n-1)th time steps. At the start point, values for n=0 are given by,

$$u(x,0) = f(x), \quad 0 \le x \le l$$

But, values for n=1 which are needed to compute u_i^2 , must be obtained from the initial-velocity condition,

$$\frac{\partial u}{\partial t}(x,0) = g(x), \quad 0 \le x \le l$$

One approach is to replace $\partial u/\partial t$ by a forward-difference approximation,

$$\frac{\partial u}{\partial t}(x_i, 0) = \frac{u_i^1 - u_i^0}{\Delta t}$$

The above equation can be rewrite as,

$$u_i^1 = u_i^0 + \Delta t \frac{\partial u}{\partial x}(x_i, 0)$$

Or,

$$u_i^1 = u_i^0 + (\Delta t) g(x_i)$$

However, this approximation has truncation error of order $O(\Delta t)$, whereas truncation error in the main equation is of order $O\left((\Delta t)^2\right)$.

Improving the Initial Approximation

$$u(x,t) = u(x,t^n) + \frac{\partial u}{\partial t}(x,t^n)(t-t^n) + \frac{\partial^2 u}{\partial t^2}(x,t^n)\frac{(t-t^n)^2}{2} +$$

$$\frac{\partial^3 u}{\partial t^3}(x,t^n)\frac{(t-t^n)^3}{6} + \ldots$$

The previous equation represents the Taylor expansion of u(x,t) about t^n. Setting $t^n = 0$, $x = x_i$ and $t = t^1$, we have,

$$u(x_i, t^1) = u(x_i, 0) + \frac{\partial u}{\partial t}(x_i, 0)\Delta t + \frac{\partial^2 u}{\partial t^2}(x_i, 0)\frac{(\Delta t)^2}{2} + O\left((\Delta t)^3\right)$$

If f'' exists, then

$$\frac{\partial^2 u}{\partial t^2}(x_i, 0) = \alpha^2 \frac{\partial^2 u}{\partial x^2}(x_i, 0) = \alpha^2 \frac{d^2 f}{dx^2}(x_i) = \alpha^2 f''(x_i)$$

and

$$u(x_i, t^1) = u(x_i, 0) + g(x_i)\Delta t + f^{(2)}(x_i)\frac{\alpha^2(\Delta t)^2}{2} + O\left((\Delta t)^3\right)$$

This produces an approximation with error $O\left((\Delta t)^3\right)$,

$$u_i^1 = f(x_i) + g(x_i)\Delta t + f^{(2)}(x_i)\frac{\alpha^2(\Delta t)^2}{2}$$

If the second derivative of *f* exists but is not readily available, we can use the equation,

$$f''(x_i) = \frac{f(x_{i+1}) - 2f(x_i) + f(x_{i-1})}{(\Delta x)^2}$$

This implies that,

$$u_i^1 = f(x_i) + g(x_i)\Delta t + \left[f(x_{i+1}) - f(x_i) + f(x_{i-1})\right]\frac{\alpha^2(\Delta t)^2}{2(\Delta x)^2}$$

Because $\lambda = (\alpha\Delta t)/\Delta x$, we ca write this as,

$$u_i^1 = (1-\lambda^2) f(x_i) + \frac{\lambda^2}{2} f(x_{i+1}) + \frac{\lambda^2}{2} f(x_{i-1}) + g(x_i)\Delta t$$

The method used for solving wave equation is **stable** only if $\alpha \frac{\Delta t}{\Delta x} \leq 1$.